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A variational approach to the problem of deep-water waves forming a circular caustic

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We consider the situation where a deep-water wavetrain approaching from infinity forms a circular caustic, is glancingly reflected at the caustic, and then propagates on out to infinity. At every point in the wavefield there are two wavetrains, the incident and reflected waves. Thus the wavefield can be treated as a slowly varying field of short-crested waves. This work generalizes that of Peregrine (1981) who considered a wavefield of incident waves only. The problem is formulated using the averaged-Lagrangian variational approach of Whitham (1974). Owing to the circular symmetry of the problem, the governing differential equations can be reduced to a set of algebraic equations at each radius. Results for the wave steepness and wavenumber are presented. These indicate that the nonlinear caustic occurs at a larger radius than does the linear caustic, and that the ray paths are no longer straight but curve away from the caustic. It is found that the slowly varying assumption is invalid at the caustic radius. To overcome this we derive, by the method of multiple scales, a modified nonlinear Schrödinger equation which is valid in this region. The solution of this equation, involving the second Painlevé transcendent, is then asymptotically matched to the slowly varying solution to provide a complete description of the wavefield.

1. Introduction

Caustics are of interest in a wavefield because of the large wave amplitudes that occur in these regions. They can arise in many physical situations such as water waves on a current, a sea-bed topography that causes waves to converge in a region, or refraction of waves due to shallowing depth. Caustics are regions in the wavefield at which linear slowly varying theory (also called linear ray theory) predicts an infinite wave amplitude. The theory is not valid in the region of the caustic as the large wave amplitude invalidates two important assumptions, that the wave properties are slowly varying and that the waves are infinitesimal. Here we derive a complete description of a wavefield of weakly nonlinear waves that involve a caustic.

A noteworthy situation that may involve caustics (see Smith 1976) occurs off the south-east coast of South Africa. Here giant waves have caused extensive damage to shipping (Mallory 1974). The Agulhas current flows down the coast at 4–5 knots and is 90–165 km wide. Ships taking advantage of this current have encountered giant waves (of the order of 15 m) when the wind produces waves travelling in opposition to the current.

Many authors have studied the phenomena of waves on currents. Smith (1976) studied the phenomenon described above, that of giant waves on the Agulhas current. He considers a deep-water wavetrain opposed by a steady irrotational current, which is reflected by a straight caustic. By a method similar to that of multiple scales, and using a perturbation expansion in a parameter related to current velocity and current gradient, he derives a modified nonlinear Schrödinger equation which describes the behaviour of the wave amplitude. He shows that the wave profile can be asymmetric and thus the wave peaks have extremely steep leading edges which can pose a danger to shipping. Smith (1975) considered the reflection of short gravity waves on a current. If the velocity of the current is below a certain critical value, the 'stopping velocity', then wave reflection does not occur. Stiassnie & Dagan (1979) have considered the situation in which the current velocity is near this critical value and found that partial reflection occurs.

In Peregrine & Smith (1979) a weakly nonlinear theory for dispersive waves near caustics is discussed. An averaged-Lagrangian method is used and results are presented for straight and curved geometries. Their main result is that nonlinearity produces caustics of two different types which they label 'R' and 'S'. The R-type caustic has a singularity in the slope of the amplitude before the position of the linear caustic is reached. This singularity may occur at a sufficiently small amplitude for the weakly nonlinear approximation to remain valid. The S-type caustic has no singularity but the wave amplitude increases rapidly past the position of the linear caustic causing the weakly nonlinear theory to become invalid. The R-type caustic is physically interpreted as a reflection of the waves (so long as the wave amplitude is small enough), while the S-type caustic is interpreted as causing the waves to break.

Schwartz (1974) considered a perturbation expansion in wave steepness to describe progressive gravity waves of permanent form. He used Padé approximants to obtain accurate results up to the highest wave. Longuet-Higgins (1975) and Cokelet (1977) used similar techniques to calculate various properties of progressive waves. These tabulated properties have been used to study the refraction, reflection and breaking of finite-amplitude water waves in various situations, using only the approximation that the nonlinear wavetrain has slowly varying properties.

Peregrine & Thomas (1979) consider finite-amplitude water waves on currents by using these tabulated properties to calculate an exact Lagrangian, which is then used to describe slowly varying finite-amplitude water waves. In one case they consider a current at 90° to the wavetrain. The waves are refracted by the current gradient and if the wavetrain becomes parallel to the current a caustic of R-type arises giving reflection. In the second case they consider a current opposed to the wavetrain. This produces an S-type caustic and represents wave breaking, although if the wavetrain propagates to the 'stopping velocity' with a small steepness reflection may occur.

Using a similar technique Ryrie & Peregrine (1982) and Peregrine & Ryrie (1983) examine finite-amplitude water waves obliquely incident onto a beach. A caustic of R-type results and near the caustic 'conjugate' solutions exist. A wavetrain approaching the beach from deep water will steepen and be refracted to propagate normal to the beach until it breaks in the lead up to the caustic singularity. However the solution of higher amplitude corresponds to a wavetrain that is refracted to propagate parallel to the beach. This behaviour is termed 'anomalous' reflection.

Peregrine (1983) considers the possibility of wave jumps between conjugate solutions. For the example described above, wave jumps can only occur for obliquely incident waves (see figure 3 of Peregrine & Ryrie 1982). He also considers the problem examined by Yue & Mei (1980), that of waves incident onto a wedge of small

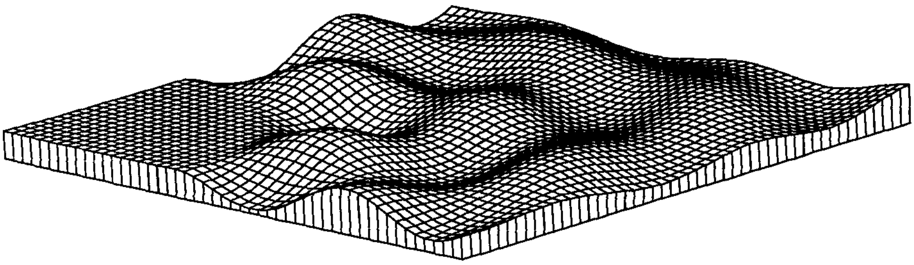

FIGURE 1. Perspective drawing of a wavefield with a circular caustic when 8 wavelengths wrap around the caustic. The centre of the caustic is in the corner on the left side of the drawing.

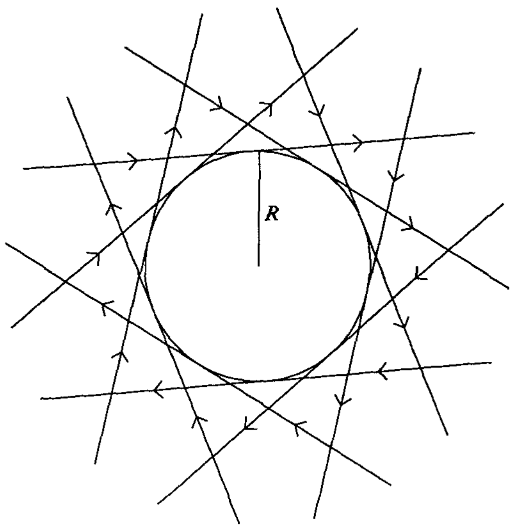

FIGURE 2. Ray paths of the linear ray solution, where the radius of the caustic is R.

angle. Wave jumps can occur for this example and he presents an analysis for a single deep-water wavetrain incident upon a wedge. His figure 3 gives the jump angle if the wave steepness is specified on both sides of the jump. Thus it is possible, where conjugate solutions exist, that one or more wave jumps will occur, causing the wavefield to be significantly modified.

Peregrine (1981) in a similar manner to Peregrine & Thomas (1979) considered waves approaching a circular caustic. Figure 1 shows a perspective drawing of a circular caustic. He characterized the problem by a caustic parameter C (figure 2 shows this situation except that Peregrine does not consider the reflected wave). The caustic is of R-type and conjugate solutions exist for sufficiently large C. This type of caustic usually indicates reflection of the wavetrain from the caustic (but if the amplitude of the wave becomes too large in the approach to the caustic then wave breaking will occur). When reflection occurs from such a circular caustic a short-crested-wave system is formed (see figure 1). However, the properties of short-crested waves (see Roberts 1983 or Marchant & Roberts 1987) can vary markedly from the progressive wave properties used by Peregrine.

Mizuguchi & Peregrine (1984) examined the question of whether waves approaching a circular caustic will produce free second-harmonic waves. Using weakly nonlinear theory they derived first- and second-order solutions in terms of

Bessel functions. They showed that at second order only forced harmonics existed, thus ruling out the possibility of second-order free harmonics.

This work extends the work of Peregrine (1981) on the circular caustic to include both the incident and reflected wavetrain. According to the linear ray theory, wave properties propagate along straight lines, thus figure 2 represents the linear ray solution to the problem we shall solve. In this solution the deep-water waves propagate towards the caustic, meet it at radius R, and then propagate away from the caustic. Far from the caustic the solution consists of two wavetrains propagating in nearly opposite directions and so looks like standing waves. Close to the caustic the two wavetrains propagate in nearly the same direction and so long-crested waves are found (Roberts & Peregrine 1983). In between, the wavetrains propagate at an angle to each other and short-crested waves occur. Our full weakly nonlinear slowly varying solution (also referred to as the nonlinear ray solution) will modify this picture by including the nonlinear interaction between the two wavetrains.

A variational approach, as developed by Whitham (1965, 1974), Ablowitz & Benney (1970) and Ablowitz (1971, 1972, 1975), is used in §2 to formulate the interaction of the two nonlinear slowly varying wavetrains. Whitham considered a singly periodic (one-phase) wavetrain and derived variational equations that describe the slowly varying properties (amplitude, wavenumber and frequency) of the wave. Ablowitz extended this to the case of wavetrains with multiple periodicities (multiple phases) and illustrated his equations by discussing various nonlinear Klein–Gordon equations. In this work the averaged Lagrangian is calculated for the case of two deep-water wavetrains (the incident and reflected waves) by substituting truncated expressions for the velocity potential ϕ and the free-surface shape η into the Lagrangian. The variational equations thus obtained are then solved exactly using an iterative scheme.

In §3 the linear ray solution is presented; this section introduces and clarifies the nature of the parameters in the later solution. In §4 we modify the model to include only the incident waves; this is the same situation that Peregrine (1981) studied using an exact Lagrangian. As expected our results reproduce his for low-amplitude waves with some variation for large-amplitude waves. In §5 the results for the full short-crested wave system formed by the presence of both the incident and the reflected waves are presented. Also considered is the angular 'Stokes drift' for this wavefield.

In §5 we note that the slowly varying assumption is invalid at the caustic radius. In §6 an equation valid in the region near the caustic is developed by the method of multiple scales. The equation, a modified nonlinear Schrödinger equation, involves the second Painlevé transcendents in its solution. Asymptotic expansions for the second Painlevé transcendent, as given in Peregrine & Smith (1979) or Miles (1978), are then used to match the near-caustic solution to the nonlinear ray solution to provide a uniformly valid description of the wavefield.

2. Variational formulation of the problem

2.1. *The variational principle for two interacting wavetrains*

We wish to describe the interaction of two nonlinear slowly varying deep-water wavetrains. To accomplish this we use Whitham's (1974) variational formulation. This formulation includes a local averaging which removes the oscillations of the wave motion from the equations leaving the slow variations in space and time of amplitude, wavenumber and frequency to be found. The averaged Lagrangian is obtained by substituting expressions for the wave motion into the Lagrangian and then averaging. The equations obtained from the averaged Lagrangian by varying the parameters in the expression then describe the slow variations of the wave properties. We consider a coordinate system where x, y are horizontal coordinates and the z-axis is vertically up. Upon assuming that the fluid is incompressible and inviscid and that the fluid motion is irrotational we consider a velocity potential ϕ and a free surface shape η given by

$$\phi = \phi(\Theta_1, \Theta_2, z), \quad \eta = \eta(\Theta_1, \Theta_2), \tag{2.1}$$

where $\Theta_1(x, y, t)$ and $\Theta_2(x, y, t)$ are two phase functions. Thus ϕ and η are periodic in both Θ_1 and Θ_2 with period 2π, and

$$\boldsymbol{k}_j = \nabla\Theta_j, \quad \omega_j = -\frac{\partial\Theta_j}{\partial t}, \quad j = 1, 2, \tag{2.2}$$

where $\boldsymbol{k}_j$ are the wavenumber vectors and ω_j are the frequencies of the two phases. For simplicity we only consider deep water, for which the Lagrangian, as proposed by Luke (1967) is

$$L = -\rho\int_{-\infty}^{\eta} (\phi_t + \tfrac{1}{2}|\nabla\phi|^2)\,\mathrm{d}z - \tfrac{1}{2}\rho g\eta^2, \tag{2.3}$$

where ρ is the density of water and g is the gravitational acceleration. We write ϕ and η in the following truncated form:

$$\begin{aligned}\phi = b_1\,\mathrm{e}^{k_1 z}\sin(\Theta_1) + b_3\,\mathrm{e}^{k_2 z}\sin(\Theta_2) + b_2\,\mathrm{e}^{2k_1 z}\sin(2\Theta_1) + b_4\,\mathrm{e}^{2k_2 z}\sin(2\Theta_2)\\ + b_5\,\mathrm{e}^{|\boldsymbol{k}_1+\boldsymbol{k}_2|z}\sin(\Theta_1+\Theta_2) + b_6\,\mathrm{e}^{|\boldsymbol{k}_1-\boldsymbol{k}_2|z}\sin(\Theta_1-\Theta_2),\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\eta = a_1\cos(\Theta_1) + a_3\cos(\Theta_2) + a_2\cos(2\Theta_1) + a_4\cos(2\Theta_2)\\ + a_5\cos(\Theta_1+\Theta_2) + a_6\cos(\Theta_1-\Theta_2).\end{aligned} \tag{2.5}$$

Normally b_1, b_3, a_1 and a_3 will be of first order in wave amplitude while the other coefficients will be of second order. The largest neglected higher harmonics in these expressions will be of third order; thus this analysis is valid only for waves that are not too high.

Upon substituting the above expressions into (2.3) and using (2.2) we find an expression for L:

$$L = L(a_i, b_i, \omega_j, \boldsymbol{k}_j, \Theta_j). \tag{2.6}$$

Assuming a_i, b_i, ω_j and $\boldsymbol{k}_j$ to be slowly varying functions of space and time and thus effectively constant over a wave period, we define the averaged Lagrangian to be

$$\bar{L}(a_i, b_i, \omega_j, \boldsymbol{k}_j) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} L\,\mathrm{d}\Theta_1\,\mathrm{d}\Theta_2. \tag{2.7}$$

$\bar{L}$ is truncated to fourth order, as the neglected harmonics first contribute at sixth order. $\bar{L}$ contains the information about the interaction of the slow variations in amplitude, wavenumber and frequency of the two wavetrains. According to Whitham (1974) the variational equations to be solved are then

$$\left.\begin{array}{l}\bar{L}_{a_i}=0\\ \bar{L}_{b_i}=0\end{array}\right\}\quad i=1,\ldots,6, \tag{2.8}$$

$$\left.\begin{array}{l}\nabla\times\boldsymbol{k}_j=0\\[6pt] \dfrac{\partial\boldsymbol{k}_j}{\partial t}+\nabla\omega_j=0\\[6pt] \dfrac{\partial\bar{L}_{\omega_j}}{\partial t}-\nabla\cdot\bar{L}_{\boldsymbol{k}_j}=0\end{array}\right\}\quad j=1,2. \tag{2.9}$$

Equations (2.8) are the results of variations of $\bar{L}$ with respect to the independent amplitude variables. The last of the equations (2.9) is the result of variations with respect to the independent phase function Θ_j. The remaining equations in (2.9) are conditions on $\boldsymbol{k}_j$ and ω_j that ensure that Θ_j exists.

2.2. *The circular caustic*

We assume that grazing reflection past a circular caustic occurs and that a short-crested wave system is formed in which the incident and reflected wavetrains have identical properties except that they travel in different directions (hence there is no loss of energy). Therefore, letting $\boldsymbol{k}_1$ be the wavenumber of the incident wave and $\boldsymbol{k}_2$ be the wavenumber of the reflected wave we deduce that

$$\boldsymbol{k}_1=-k_r\boldsymbol{e}_r+k_\theta\boldsymbol{e}_\theta,\quad \boldsymbol{k}_2=+k_r\boldsymbol{e}_r+k_\theta\boldsymbol{e}_\theta, \tag{2.10}$$

and
$$a_1=a_3,\quad a_2=a_4,\quad b_1=b_3,\quad b_2=b_4,\quad \omega_1=\omega_2, \tag{2.11}$$

where k_r and k_θ represent the radial and angular wavenumber components respectively and $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ represent unit vectors in the direction of increasing r and the direction of increasing θ respectively. Figure 2 shows this situation where R represents the radius of the caustic (according to the linear ray theory of §3). We assume that the properties of the wavefield are steady ($\partial/\partial t=0$) and axisymmetric ($\partial/\partial\theta=0$); hence the solution depends only on r. Thus (2.8) can be simplified, and (2.9) can be integrated with respect to r to give

$$\left.\begin{array}{l}\bar{L}_{a_1}=0,\quad \bar{L}_{a_2}=0,\quad \bar{L}_{a_5}=0,\quad \bar{L}_{a_6}=0,\\ \bar{L}_{b_1}=0,\quad \bar{L}_{b_2}=0,\quad \bar{L}_{b_5}=0,\quad \bar{L}_{b_6}=0,\\ k_\theta=R/r,\quad \omega_1=\omega_2=\omega,\\ \bar{L}_{k_r}=A/r,\end{array}\right\} \tag{2.12}$$

where R, ω and A are integration constants.

To calculate $\bar{L}$ in a relatively simple form (see the Appendix) we let ψ be half the angle between $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, and $k=k_1=k_2$. Then we may write

$$k_r=k\alpha,\quad k_\theta=k\beta, \tag{2.13}$$

where
$$\alpha=\sin(\psi),\quad \beta=\cos(\psi), \tag{2.14}$$

and obtain
$$\bar{L}=\bar{L}(a_i,b_i,\omega,k,\alpha,\beta). \tag{2.15}$$

The last equation in (2.12) then becomes

$$\bar{L}_k\left(\frac{k_r}{k}\right)-\bar{L}_\beta\left(\frac{k_\theta k_r}{k^3}\right)+\bar{L}_\alpha\left(\frac{k_\theta^2}{k^3}\right)=\frac{A}{r}. \tag{2.16}$$

For simplicity we non-dimensionalize the problem with respect to the reference length $2\pi/k_0$ and the reference time $(gk_0)^{-\frac{1}{2}}$, where k_0 is the wavenumber of the waves at infinity; thus $\omega = 1$. Further, we emphasize that the non-dimensional wavelength of the incident waves is 2π and so the slowly varying assumption is valid only if variations in wave properties occur on a lengthscale that is larger than this.

The equations obtained, (2.12), are a set of algebraic equations for the short-crested wavefield formed by the incident and reflected waves which must be solved at each radius r (the details of the equations can be found in the Appendix). The equations are too complicated to find an explicit analytic solution and so are solved numerically for each radius using a variation of Newton's method.

3. The linear ray solution

We first consider the linear ray solution to illustrate clearly our solution technique and also to highlight the changes that occur when nonlinearity is introduced. To obtain the ray solution to this problem, as seen in figure 2, we consider just the lowest-order terms in $\bar{L}$ (i.e. the terms of second order). Thus we neglect the nonlinear interaction terms with coefficients a_2, a_4, a_5, a_6, b_2, b_4, b_5 and b_6. Then we obtain

$$\left.\begin{aligned}&\bar{L} = a_1^2+kb_1^2-2\omega b_1 a_1,\\ &\bar{L}_{a_1} = 2a_1-2\omega b_1, \quad \bar{L}_{b_1} = 2kb_1-2\omega a_1,\\ &\bar{L}_k = b_1^2, \quad \bar{L}_\alpha = 0, \quad \bar{L}_\beta = 0.\end{aligned}\right\} \tag{3.1}$$

Hence the solution of (2.12) is

$$\left.\begin{aligned}&\omega = 1, \quad k_\theta = R/r, \quad k_r = [1-(R/r)^2]^{\frac{1}{2}},\\ &a_1^2 = b_1^2 = \frac{A}{r[1-(R/r)^2]^{\frac{1}{2}}}.\end{aligned}\right\} \tag{3.2}$$

The amplitude becomes infinite, like $(r-R)^{-\frac{1}{4}}$ as $r\rightarrow R$; hence the integration constant R determines the radius of the circular caustic. The integration constant A determines the overall amplitude of the wavefield; this is because the energy density of the waves is proportional to $a_1^2 \sim A/r$ as $r\rightarrow\infty$. The wave amplitude is taken to be zero inside the circular caustic, that is in $r < R$.

The caustic radius R is related to the number of waves around the caustic which at any radius is given by rk_θ. From (3.2), and also for the full problem (2.12), this is just R. Thus R can also be interpreted as the number of waves around the caustic. The solution is required to be periodic so R must be an integer to obtain a physically realizable solution.

If a ray path is described by $r(\theta)$ then its tangent is in the direction of the local wavenumber, and so

$$\frac{\mathrm{d}r}{\mathrm{d}\theta}=\frac{rk_r}{k_\theta}. \tag{3.3}$$

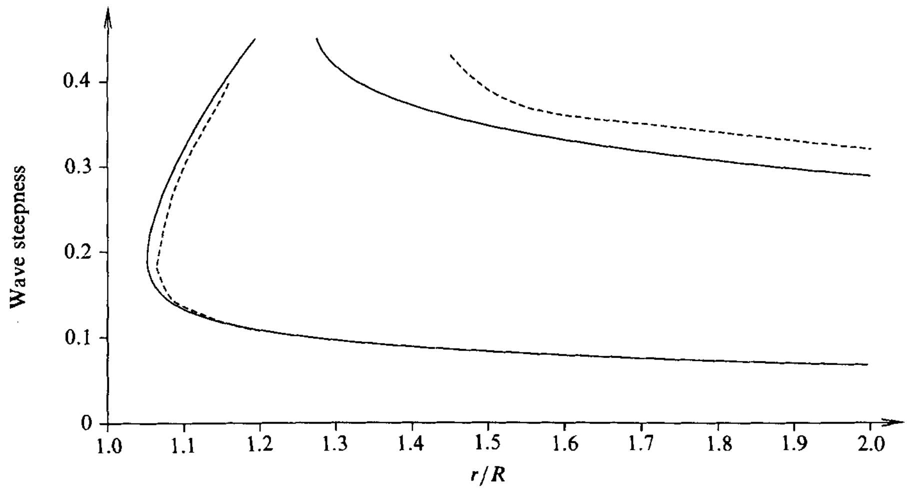

FIGURE 3. Wave steepness $a_1 k$ versus radius r/R for a purely incident wavefield. Compared are the present theory (——) and Peregrine's (1981) results (----) for $A/R = 0.008$ (lower curves) and $A/R = 0.2$ (upper curves).

With k_r and k_θ as given by (3.2) the solution of (3.3) is

$$r = \frac{R}{\sin(\theta-\gamma)}, \tag{3.4}$$

where γ is an arbitrary constant of integration. As expected, the ray path is seen to be a straight line at an angle γ to the x-axis and passing a distance R from the origin (see figure 2).

4. Solutions with only an incident wavefield

If we let the free-surface shape η and the velocity potential ϕ be a function of one phase function only, that is $\eta = \eta(\Theta_1)$ and $\phi = \phi(\Theta_1, z)$, then our model will be the same as Peregrine's (1981) model in that only the incident wavetrain is considered. However, our model is only weakly nonlinear and hence is more restricted than Peregrine's. The solutions we examine here are relevant both as a check of our method and also if the incident waves break somewhere in the wavefield.

Peregrine's caustic parameter C is related to our amplitude parameter A and the caustic radius R by the relation

$$\frac{A}{R} = \frac{4}{C}. \tag{4.1}$$

Figure 3 shows our results for wave steepness $a_1 k$ versus r/R the radius parameter, for $A/R = 0.008$ and 0.2 compared against Peregrine's results for these values. For $A/R = 0.008$ they compare well but for $A/R = 0.2$ there is some variation and we obtain about a 10% difference in the radius at which the caustic singularity occurs. This is due to the fact that Peregrine used an exact Lagrangian which is more accurate at large wave steepnesses than our weakly nonlinear Lagrangian. The caustic is of an R-type and hence represents reflection as long as the wave steepness is small enough in the approach to the caustic. Where reflection does occur the model

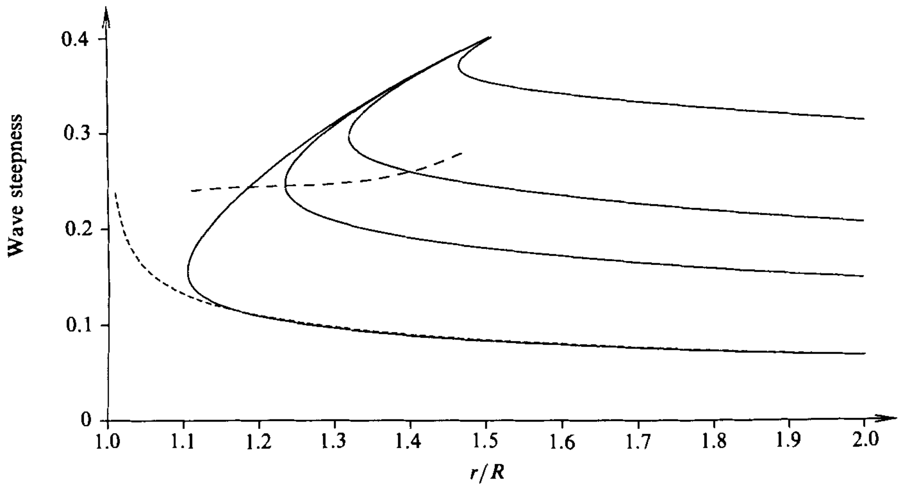

FIGURE 4. Wave steepness $a_1 k$ versus radius r/R for the combined incident and reflected wavefields. Shown is the present theory (——) for $A/R = 0.008$, 0.04, 0.08 and 0.2 (lower to upper curves respectively). Shown for comparison is the linear theory (----) for $A/R = 0.008$. Also given are estimates of the maximum steepness of the short-crested wavefield at the caustic radius (— — —).

containing both the incident and reflected wavetrains, of the next section, is appropriate.

It can be seen that 'conjugate' solutions exist near the caustic. The lower branch of solutions corresponds to the linear solution in the limit as $A/R \to 0$. The higher branch of solutions, which exists over a finite range of r/R, are approximately waves with radial crests which move in a purely angular direction ($k_r = 0$) and have wave steepness increasing with r/R. In the limit $A/R \to 0$ (by letting $k_r \to 0$, not $a_1 \to 0$ as for linear waves) the higher branch of solutions approaches the radial solutions of Peregrine (1981, §5).

5. The combined incident and reflected wavefields

5.1. *Discussion of results*

In this section we return to the model that includes both the incident and reflected wavetrains. Figure 4 shows the wave steepness $a_1 k$ versus r/R for $A/R = 0.008$, 0.04, 0.08 and 0.2. In each case the results are qualitatively similar to Peregrine's, i.e. a singularity occurs at finite wave steepness before the linear caustic radius is reached. However, there are quantitative differences; for example, our theory predicts that the singularity occurs at a larger radius than in Peregrine's theory (which only contains the incident wave).

Also included in figure 4 are estimates of the maximum steepness of short-crested waves (from Roberts 1983) for the short-crested waves found at the caustic. Since the wavefield varies from near standing waves far from the caustic to long-crested waves near the caustic (which have different maximum steepnesses) the possibility exists that the waves break somewhere in the wavefield before the caustic is reached even though the wavefield has a wave steepness below the maximum wave steepness at the caustic. This occurrence is unlikely though: the maximum wave steepness is smallest in the long-crested region of the short-crested-wave parameter range. Hence if a wavetrain has not reached maximum wave steepness at the caustic it is unlikely to have reached it elsewhere in the wavefield. Figure 4 shows that for $A/R = 0.008$ and 0.04 the wavefield has everywhere a wave steepness below the maximum wave steepness. In this case the singularity is interpreted as the nonlinear caustic at which reflection occurs and the two-phase analysis presented here must be used to represent the resultant short-crested wave system.

For $A/R = 0.08$ and 0.2 the caustic singularity occurs at a wave steepness greater than the maximum short-crested wave steepness, thus wave breaking must occur somewhere in the wavefield. This is inconsistent with the assumption that no energy is lost as the waves propagate. In this case Peregrine's model, as discussed in §4, is more appropriate.

Since conjugate solutions exist, wave jumps can occur. Hence the possibility exists that owing to wave jumps the wavefield will be modified and the problem we are describing will not occur. For example, the incident wavetrain would be only partially reflected at a wave jump, thus violating our assumption of perfect reflection. However, such a wave jump is unlikely to occur (see Peregrine 1983, p. 444). Behind a wave jump the waves would travel in a purely angular direction (the limit as $k_r \to 0$ of the second solution branch). This is expected by analogy with Yue & Mei (1980) where behind the wave jump the waves travel parallel with the reflecting wedge. But on the other side of the wave jump the waves would have a radial component. Hence the creation of a steady wave jump is unlikely. It should be noted however that unsteady wave jumps would be possible, dependent on how the caustic is created.

One point of interest is the nonlinear ray paths of the incident and reflected waves. To obtain a ray path $r(\theta)$, we solve the differential equation (3.3) numerically. At infinity the ray path is virtually straight, but as the ray approaches the caustic it is bent away from the caustic. At the nonlinear caustic the ray is reflected, and propagates outwards to infinity (along a ray that is a reflection of the incident wave), thus setting up a short-crested wave system. Note that the reflection is 'sharp', that is the radial wavenumber is not zero at the caustic. Thus the assumption that the wave properties are slowly varying is invalid at this radius as the radial wavenumber is rapidly changing (as is the wave amplitude also).

The solution to the problem of waves approaching a circular caustic can also be described in terms of Bessel functions (see Peregrine 1981 or Mizuguchi & Peregrine 1984). Figure 5 shows the surface elevation η versus r/R for the linear Bessel-function solution. $A/R = 0.06$ and there are 40 waves around the caustic. For comparison, the wave steepness $a_1 k$ for our weakly nonlinear model is also shown. As can be seen our weakly nonlinear results form an 'envelope' over the linear Bessel-function solution. Far from the caustic our weakly nonlinear results form a near-exact envelope over the linear Bessel-function solution but as the caustic is approached there is some deviation and our weakly nonlinear envelope is of lower steepness than the linear Bessel-function solution. Also the weakly nonlinear caustic singularity occurs before the radius at which the linear Bessel function reaches its maximum surface elevation.

5.2. *The 'Stokes drift' of the wavefield*

According to linear theory a fluid particle when being acted upon by a sinusoidal wave in deep water will trace out a circular path, the time taken to complete a cycle being the period $2\pi/\omega$. However, calculating a fluid particle's path to higher orders results in a correction to that path. In particular the mean horizontal velocity is no longer zero, this quantity being known as the 'Stokes drift'.

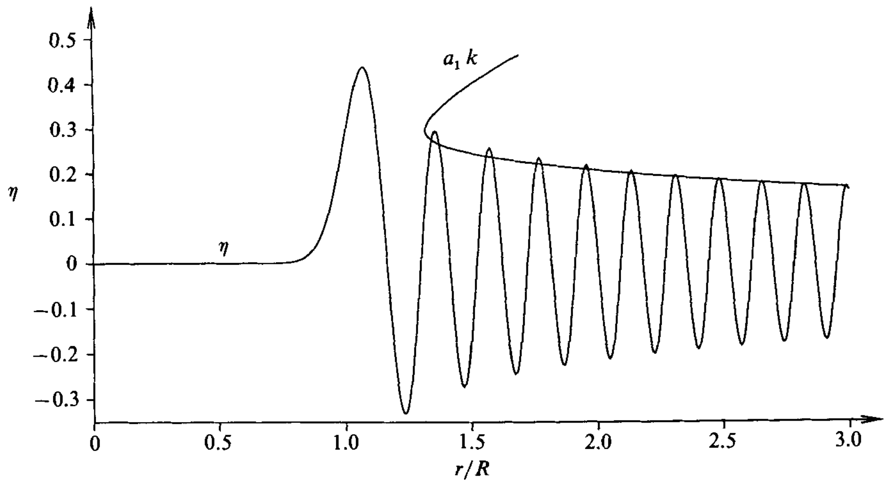

FIGURE 5. The surface elevation η versus r/R for the linear Bessel-function solution. $A/R = 0.06$ and there are 40 waves around the caustic. Drawn above this is the wave steepness $a_1 k$ versus radius r/R for the combined incident and reflected wavefields.

For the circular caustic problem under discussion we shall consider the angular Stokes drift and calculate the lowest-order term (of order two) of this quantity (see Lighthill 1978, p. 280). To first order,

$$\phi = b_1 \, \mathrm{e}^{kz} \sin(\Theta_1) + b_1 \, \mathrm{e}^{kz} \sin(\Theta_2). \tag{5.1}$$

Now a fluid particle at depth $z = d$ in the undisturbed fluid will be at depth

$$z = \eta_d = d - kb_1 \, \mathrm{e}^{kd} \cos(\Theta_1) - kb_1 \, \mathrm{e}^{kd} \cos(\Theta_2) \tag{5.2}$$

at time t (to first order). The average momentum over a wave period in the angular direction of the fluid particles between the undisturbed level $z = d$ and $z = -\infty$ is

$$M = \frac{\rho}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \int_{-\infty}^{\eta_d} \phi_\theta \, \mathrm{d}z \, \mathrm{d}\Theta_1 \, \mathrm{d}\Theta_2. \tag{5.3}$$

Integrating (5.3) gives

$$M = \rho k_\theta b_1^2 \, \mathrm{e}^{2kd} \tag{5.4}$$

to second order. Thus by differentiating (5.4) with respect to d we obtain as the second-order contribution to the angular Stokes drift at the depth d

$$2k_\theta k b_1^2 \, \mathrm{e}^{2kd}. \tag{5.5}$$

In this calculation the first neglected terms will be of fourth order so this result is valid for waves not too high. However, it is worth noting that the second-order terms in (2.4) (only b_2, b_4 and b_5 as $b_6 = 0$) decay like e^{2kd} as $d \to -\infty$. So the result (5.5) is valid for estimating the Stokes drift sufficiently deep in the fluid regardless of wave steepness.

Figure 6 shows the angular Stokes drift versus r/R for $d = -0.1$, -0.2, -0.4 and -0.8 when $A/R = 0.04$. The angular Stokes drift is shown for the combined incident and reflected wavefield in the linear and in the weakly nonlinear theory, and it increases as the caustic is approached and also is larger near the surface of the fluid. As can be seen from the diagram the weakly nonlinear theory predicts a significantly larger angular Stokes drift than linear theory.

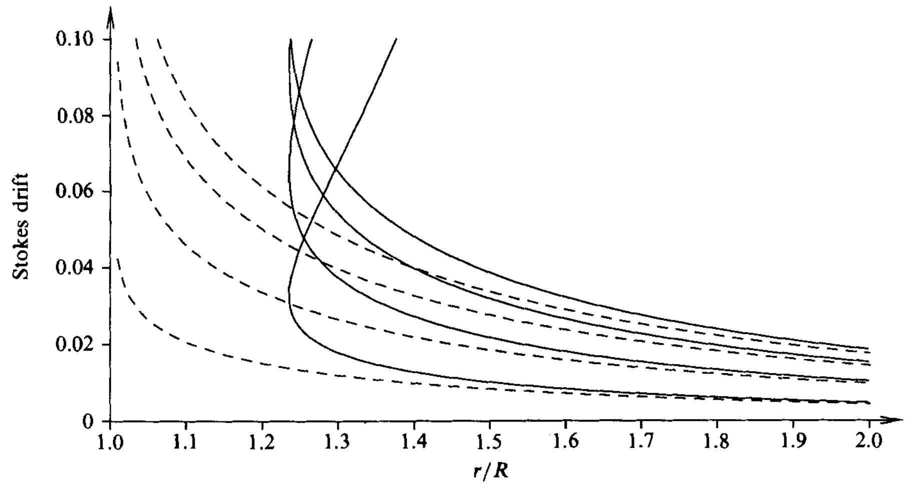

FIGURE 6. The angular Stokes drift versus radius r/R for the combined incident and reflected wavefields. Shown is the present theory (——) for $A/R = 0.04$ at depths $d = -0.1$, -0.2, -0.4 and -0.8 (upper to lower curves respectively). Shown for comparison is the linear theory (----).

6. The region near the nonlinear caustic

6.1. *The modified nonlinear Schrödinger equation*

In the previous section it was found that one of the assumptions, that the wave properties be slowly varying, was invalid near the caustic. Here we derive an equation that is valid in this region, and the solutions will then be matched onto the nonlinear ray solution of §5. We derive this equation by the method of multiple scales in a manner similar to Smith's (1976) derivation of his model equation for the case of a wavetrain approaching a straight caustic. Since the waves near the caustic look like a long-crested wavefield the derived equation is a nonlinear Schrödinger equation, but modified by the presence of the caustic.

In deriving the nonlinear Schrödinger equation we follow the usual technique as used by Smith (1976) or Yue & Mei (1980). Owing to the circular symmetry we write Laplace's equation and the usual boundary conditions in cylindrical coordinates (r, θ, z) and to investigate the region near the caustic we define new coordinates (ρ, ξ) by

$$r = R+\rho, \quad \theta = \xi/R. \tag{6.1}$$

We envisage that R is approximately the radius of the caustic and is assumed large; ρ measures the distance from the caustic and hence ρ/R is small. The coordinate ξ measures distance around the caustic. Upon substituting (6.1) into Laplace's equation and the boundary conditions and expanding in the parameter ρ/R, first-order equations in ρ/R are obtained.

We define a small parameter ϵ by $\epsilon^3 = 1/R$ and introduce the new long-space, slow-time variables

$$X = \epsilon\rho, \quad Y = \epsilon(\xi-ct), \quad T = \epsilon^2 t, \tag{6.2}$$

and $\chi = \xi - t$, where c is the group velocity. The conditions at infinite radius require that there are precisely R waves around the caustic and that the frequency $\omega = 1$. Thus χ will serve as the phase function of the waves around the caustic.

We assume a perturbation expansion of the form

$$\left.\begin{aligned}\phi &= \epsilon\phi_1(X, Y, T, \chi, z) + \epsilon^2\phi_2(X, Y, T, \chi, z) + \dots,\\ \eta &= \epsilon\eta_1(X, Y, T, \chi) + \epsilon^2\eta_2(X, Y, T, \chi) + \dots,\end{aligned}\right\} \tag{6.3}$$

and write the leading-order equations in terms of the new variables (6.2) and the phase function χ. The perturbation expansions (6.3) are substituted into the leading-order equations, terms of the same order are grouped together, and then the resultant equations are solved in succession.

At first order the solution is

$$\phi_1 = -\mathrm{i}\mathscr{A}\,\mathrm{e}^{\mathrm{i}\chi + kz} + \text{c.c.}, \quad \eta_1 = \mathscr{A}\,\mathrm{e}^{\mathrm{i}\chi} + \text{c.c.}, \tag{6.4}$$

where $\mathscr{A} = \mathscr{A}(X, Y, T)$ is an amplitude coefficient. The second-order solution is

$$\phi_2 = -\mathscr{A}_Y z\,\mathrm{e}^{\mathrm{i}\chi + kz} + \text{c.c.}, \quad \eta_2 = -\mathrm{i}\tfrac{1}{2}\mathscr{A}_Y\,\mathrm{e}^{\mathrm{i}\chi} + \mathscr{A}^2\,\mathrm{e}^{\mathrm{i}2\chi} + \text{c.c.}. \tag{6.5}$$

At all orders we require that the coefficients of the secular term $\mathrm{e}^{\mathrm{i}\chi}$ on the right-hand side of the equations be zero. At third order this condition gives the equation

$$\mathrm{i}\mathscr{A}_T + \tfrac{1}{2}\mathscr{A}_{XX} - \tfrac{1}{4}\mathscr{A}_{YY} + X\mathscr{A} - 4|\mathscr{A}|^2\mathscr{A} = 0. \tag{6.6}$$

This is the modified nonlinear Schrödinger equation; its solutions involve Painlevé transcendents of the second kind.

We consider steady solutions for the wavefield properties ($\mathscr{A}_T = 0$), and also recognize that there will be no dependence upon Y ($\mathscr{A}_{YY} = 0$) as there are always R waves around the caustic. By reverting to the original unscaled variables and defining the unscaled amplitude $a(\rho) = \epsilon\mathscr{A}$ we obtain the equation

$$F_{z'z'} - z'F - 2|F|^2F = 0, \tag{6.7}$$

where $$\rho = -(\tfrac{1}{2}R)^{\frac{1}{3}}z', \quad a(\rho) = \frac{1}{2^{\frac{2}{3}}R^{\frac{1}{3}}}F(z').$$

The solutions of (6.7) are Painlevé transcendents of the second kind (see Miles 1978). Following Miles' notation the relevant solutions are the one-parameter family $F = F(z', a_-)$. When $a_- = 0$ the solution of the resultant linear equation is an Airy function. In fact Painlevé transcendents are qualitatively similar to Airy functions with the transition from sinusoidal to exponential behaviour displaced from $z' = 0$ to negative z' (i.e. larger ρ) as the nonlinearity a_- increases.

6.2. *Matching the solutions*

The modified nonlinear Schrödinger equation is valid in a region near the caustic. The nonlinear ray solution is valid nearly to the caustic radius. Our aim here is to find a region in which both solutions are valid and then asymptotically match them to obtain a uniformly valid description of the wavefield.

The surface elevation of the nonlinear ray solution is

$$\eta(r, \theta, t) = 2a_1 \cos\left(\frac{\Theta_1 - \Theta_2}{2}\right)\cos\left(\frac{\Theta_1 + \Theta_2}{2}\right) + 2a_2 \cos(\Theta_1 - \Theta_2)\cos(\Theta_1 + \Theta_2)$$
$$+ a_5 \cos(\Theta_1 + \Theta_2) + a_6 \cos(\Theta_1 - \Theta_2), \tag{6.8}$$

where all variables have been previously defined. Now $\nabla\Theta_j = \boldsymbol{k}_j$, where the $\boldsymbol{k}_j$ are given by (2.10). This implies that

$$\Theta_1 = -\int k_r\,\mathrm{d}r + R\theta - t, \quad \Theta_2 = \int k_r\,\mathrm{d}r + R\theta - t. \tag{6.9}$$

We now consider the nonlinear ray solution as it approaches the caustic for small amplitudes. To concentrate on this approach we introduce a small parameter δ which relates the magnitudes of various quantities by

$$a_1, b_1 = O(\delta), \quad \frac{\rho}{R} = O(\delta), \quad k_r = O(\delta^{\frac{1}{2}}) \quad \text{as } \delta \to 0. \tag{6.10}$$

By considering the variational equations (2.12) and equations (2.13) we find the following leading-order expressions:

$$\left.\begin{aligned} \beta \sim k_\theta \sim 1-\frac{\rho}{R}, \quad \alpha = O(\delta^{\frac{1}{2}}), \\ a_2 \sim \tfrac{1}{2}a_1^2, \quad a_5 \sim a_1^2, \quad a_6, b_2, b_5, b_6 \sim 0. \end{aligned}\right\} \tag{6.11}$$

and

$$\omega^2 \sim k(1+3k^2a_1^2), \quad k_r a_1^2 \sim \frac{A}{R}. \tag{6.12}$$

The first of equations (6.12) implies

$$k_r^2 \sim 2\frac{\rho}{R} - 6a_1^2 \tag{6.13}$$

and substituting for k_r from the second of equations (6.12) gives

$$a_1^6 - \frac{\rho}{3R}a_1^4 + \frac{1}{6}\left(\frac{A}{R}\right)^2 = 0. \tag{6.14}$$

This mixed-order equation is equivalent to (3.9) in Peregrine & Smith (1979) and describes the near-pitchfork structure near the caustic (see figure 1*a* in the above-mentioned paper). If we consider just the lowest-order terms in (6.14) ($a_1^6 = O(\delta^6)$ while the other terms are $O(\delta^5)$) we obtain the leading-order expressions

$$\left.\begin{aligned} a_1^2 &\sim \frac{1}{\sqrt{2}}\frac{(A/R)}{(\rho/R)^{\frac{1}{2}}}, \\ k_r &\sim \sqrt{2}(\rho/R)^{\frac{1}{2}} - \frac{3}{2}\frac{(A/R)}{(\rho/R)}. \end{aligned}\right\} \tag{6.15}$$

This expression for a_1 just describes the same growth in amplitude approaching the caustic as in linear theory. The expression for the radial wavenumber k_r is the same as in linear theory but is modified by the nonlinear effects.

The surface elevation obtained from the nonlinear ray solution for small amplitudes approaching the caustic is then

$$\eta(r,\theta,t) = 2a_1\cos\left(\int k_r\,\mathrm{d}r\right)\cos(R\theta - t) + 2a_1^2\cos^2\left(\int k_r\,\mathrm{d}r\right)\cos(2R\theta - 2t), \tag{6.16}$$

where a_1 and k_r are given by (6.15).

The surface elevation from the nonlinear Schrödinger equation is obtained by substituting the expressions for η_1 and η_2 from (6.4) and (6.5) into the perturbation expansion (6.3) for $\eta(r,\theta,t)$ and replacing $\mathscr{A}$ by a/ϵ. We obtain

$$\eta = 2a(\rho)\cos(R\theta-t)+2a(\rho)^2\cos(2R\theta-2t), \tag{6.17}$$

where $a(\rho)$ comes from the solution to (6.7) which has the asymptotic form (equation (6.12) from Miles 1978)

$$F \sim \frac{a_-}{|z'|^{\frac{1}{4}}}\cos\left(\tfrac{2}{3}|z'|^{\frac{3}{2}}-\tfrac{1}{2}a_-^2[\ln(|z'|^{\frac{3}{2}}/a_-^2)+3.366]\right) \quad \text{as}\, z'\rightarrow-\infty. \tag{6.18}$$

This is valid for

$$-z' \gg a_-^{\frac{4}{3}} \gg 1, \tag{6.19}$$

as $-(\pi a_-)^{\frac{4}{3}}$ is the approximate position of the maximum amplitude for the Painlevé transcendent for large a_- (from Miles 1978). Equation (6.7) was derived using ρ/R as a small parameter. Rearranging (6.19) gives

$$\left(\frac{a_-^2}{R}\right)^{\frac{2}{3}} \ll \frac{\rho}{R} \ll 1 \tag{6.20}$$

as a region to match the solutions for large R. Equation (6.20) represents a region before the caustic is reached, so the solutions are matched in the approach to the caustic where the nonlinear ray solution is still valid.

To match (6.17) in the region (6.20) to the nonlinear ray solution (6.16) we observe that the angular and time dependences are identical, thus we only need to consider the radial structure of the solution. First the constant in (6.18), which represents a phase of the incident and reflected waves, appears implicitly in the nonlinear ray solution (6.16) as a constant of integration. This occurs because the ray solution is calculated using a Lagrangian where all phase information is averaged out, while the nonlinear Schrödinger equation retains phase information. Differentiating and considering the radial wavenumbers k_r we find that the implicit radial wavenumber in (6.18) is

$$k_r = \surd 2\left(\frac{\rho}{R}\right)^{\frac{1}{2}}-\frac{3a_-^2}{4R}\left(\frac{\rho}{R}\right). \tag{6.21}$$

The implicit amplitude factor in (6.18) is

$$\frac{a_-}{|z'|^{\frac{1}{4}}} = \frac{a_-}{2^{\frac{3}{4}}R^{\frac{1}{2}}(\rho/R)^{\frac{1}{4}}}. \tag{6.22}$$

Hence to match (6.16) and (6.17) we compare (6.21) and (6.22) with (6.15) and find they are identical if the parameter of the second Painlevé transcendent is

$$a_- = \surd 2A^{\frac{1}{2}}. \tag{6.23}$$

The composite description of the wavefield is then the nonlinear ray solution away from the caustic region, and the solution of the modified nonlinear Schrödinger equation in the caustic region. These two solutions are matched in the region given by (6.20). There is no singularity in this solution as the second Painlevé transcendent makes a smooth transition across the caustic from a region of no waves to a region of nonlinear waves. In the matching region the solution of the modified nonlinear Schrödinger equation, given by (6.18), can be interpreted as a ray solution and therefore gives the same ray paths as the nonlinear ray solution. Hence we can confirm the nonlinear ray solution to be correct, even though the assumption that the wave properties are slowly varying is invalid at the caustic.

Appendix

The averaged Lagrangian for the circular-caustic problem was calculated with the aid of CAMAL, an algebraic manipulation package, and is

$$\begin{aligned}\bar{L} = {} & a_1^2+kb_1^2-2\omega b_1 a_1+\tfrac{1}{2}a_5^2+\tfrac{1}{2}a_6^2+a_2^2+2kb_2^2 \\ & -2\omega b_5 a_5-4\omega b_2 a_2+k\alpha b_6^2+k\beta b_5^2+4k^2 b_1 b_2 a_1-k^2 b_1^2 a_5 \\ & -2\omega k b_2 a_1^2-\omega k b_1 a_1 a_5-\omega k b_1 a_1 a_6-\omega k b_1 a_2 a_1+2k^2\beta b_1 b_5 a_1 \\ & -2\omega k\beta b_5 a_1^2+2k^2\beta^2 b_1 b_5 a_1+k^2\beta^2 b_1^2 a_5+k^2\beta^2 b_1^2 a_6+k^3 b_1^2 a_1^2 \\ & -\tfrac{3}{4}\omega k^2 b_1 a_1^3+2k^3\beta^2 b_1^2 a_1^2.\end{aligned}$$

The variations of $\bar{L}$ with the coefficients of the free-surface shape η are

$$\begin{aligned}\bar{L}_{a_1} = {} & 2a_1-\omega 2b_1+4k^2 b_1 b_2-4\omega k b_2 a_1-\omega k b_1 a_5-\omega k b_1 a_6 \\ & -\omega k b_1 a_2-k^2 a b_1 b_6+2k^2\beta b_1 b_5-2\omega k\beta b_5 a_1-2k^2\alpha^2 b_1 b_6 \\ & +2k^2\beta^2 b_1 b_5+2k^3 b_1^2 a_1-\tfrac{9}{4}\omega k^2 b_1 a_1^2+4k^3\beta^2 b_1^2 a_1, \\ \bar{L}_{a_2} = {} & 2a_2-4\omega b_2-\omega k b_1 a_1, \\ \bar{L}_{a_5} = {} & a_5-2\omega b_5-k^2 b_1^2-\omega k b_1 a_1+k^2\beta^2 b_1^2, \\ \bar{L}_{a_6} = {} & a_6-\omega k b_1 a_1+k^2\beta^2 b_1^2.\end{aligned}$$

The variations of $\bar{L}$ with the coefficients of the velocity potential ϕ are

$$\begin{aligned}\bar{L}_{b_1} = {} & 2kb_1-2\omega a_1+4k^2 b_2 a_1-2k^2 b_1 a_5-\omega k a_1 a_5-\omega k a_1 a_6 \\ & -\omega k a_2 a_1+2k^2\alpha b_6 a_1+2k^2\beta b_5 a_1+2k^2\alpha^2 b_6 a_1+2k^2\beta^2 b_5 a_1 \\ & +2k^2\beta^2 b_1 a_5+2k^2\beta^2 b_1 a_6+2k^3 b_1 a_1^2-\tfrac{3}{4}\omega k^2 a_1^3+4k^3\beta^2 b_1 a_1^2, \\ \bar{L}_{b_2} = {} & 4kb_2-2\omega a_2+4k^2 b_1 a_1-2\omega k a_1^2, \\ \bar{L}_{b_5} = {} & -2\omega a_5+2k\beta b_5+2k^2\beta b_1 a_1-2\omega k\beta a_1^2+2k^2\beta^2 b_1 a_1, \\ \bar{L}_{b_6} = {} & 2k\alpha b_6.\end{aligned}$$

The variations of $\bar{L}$ with respect to the wavenumber involves

$$\begin{aligned}\bar{L}_k = {} & 2b_2^2+b_1^2+\alpha b_6^2+\beta b_5^2+8kb_1 b_2 a_1-2kb_1^2 a_5-2\omega b_2 a_1^2 \\ & -\omega b_1 a_1 a_5-\omega b_1 a_1 a_6-\omega b_1 a_2 a_1+4k\beta b_1 b_5 a_1-2\omega\beta b_5 a_1^2 \\ & +4k\beta^2 b_1 b_5 a_1+2k\beta^2 b_1^2 a_5+2k\beta^2 b_1^2 a_6+3k^2 b_1^2 a_1^2-\tfrac{3}{2}\omega k b_1 a_1^3+6k^2\beta^2 b_1^2 a_1^2, \\ \bar{L}_\beta = {} & kb_5^2+2k^2 b_1 b_5 a_1-2\omega k b_5 a_1^2+4k^2\beta b_1 b_5 a_1+2k^2\beta b_1^2 a_5+2k^2\beta b_1^2 a_6+4k^3\beta b_1^2 a_1^2, \\ \bar{L}_\alpha = {} & kb_6^2.\end{aligned}$$

REFERENCES

Ablowitz, M. J. 1971 Applications of slowly varying nonlinear dispersive wave theory. *Stud. Appl. Maths* **50**, 329–344.

Ablowitz, M. J. 1972 Approximate methods for obtaining multi-phase modes in nonlinear dispersive wave problems. *Stud. Appl. Maths* **51**, 17–55.

Ablowitz, M. J. 1975 A note on resonance and nonlinear dispersive waves. *Stud. Appl. Maths* **54**, 61–70.

Ablowitz, M. J. & Benney, D. J. 1970 The evolution of multi-phase modes for nonlinear dispersive waves. *Stud. Appl. Maths* **47**, 225–238.

Cokelet, E. D. 1977 Steep gravity waves in water of arbitrary uniform depth. *Phil. Trans. R. Soc. Lond.* A **286**, 183–230.

Lighthill, M. J. 1978 *Waves in Fluids*. Cambridge University Press.

Longuet-Higgins, M. S. 1975 Integral properties of periodic waves of finite amplitude. *Proc. R. Soc. Lond.* A **342**, 157–174.

Luke, J. C. 1967 A variational principle for a fluid with a free surface. *J. Fluid Mech.* **27**, 395–397.

Mallory, J. K. 1974 Abnormal waves on the south east coast of South Africa. *Intl Hydrog. Rev.* **51**, 99–129.

Marchant, T. R. & Roberts, A. J. 1987 Properties of short-crested waves in water of finite depth. *J. Austral. Math. Soc.* B **29**, 103–125.

Miles, J. W. 1978 On the second Painlevé transcendent. *Proc. R. Soc. Lond.* A **361**, 277–291.

Mizuguchi, M. & Peregrine, D. H. 1984 Second-order theory of waves approaching circular caustics. *Rep.* AM-84-02. University of Bristol.

Peregrine, D. H. 1981 Refraction of finite-amplitude water waves: deep-water waves approaching circular caustics. *J. Fluid Mech.* **109**, 63–74.

Peregrine, D. H. 1983 Wave jumps and caustics in the propagation of finite-amplitude water waves. *J. Fluid Mech.* **136**, 435–452.

Peregrine, D. H. & Ryrie, S. 1983 Anomalous reflection and conjugate solutions of finite-amplitude water waves. *J. Fluid Mech.* **134**, 91–101.

Peregrine, D. H. & Smith, R. 1979 Nonlinear effects upon waves near caustics. *Phil. Trans. R. Soc. Lond.* A **292**, 341–370.

Peregrine, D. H. & Thomas, G. P. 1979 Finite amplitude deep water waves on currents. *Phil. Trans. R. Soc. Lond.* A **292**, 371–390.

Roberts, A. J. 1983 Highly nonlinear short-crested waves. *J. Fluid Mech.* **135**, 301–321.

Roberts, A. J. & Peregrine, D. H. 1983 Notes on long-crested water waves. *J. Fluid Mech.* **135**, 323–335.

Ryrie, S. & Peregrine, D. H. 1982 Refraction of finite-amplitude water waves obliquely incident on a uniform beach. *J. Fluid Mech.* **115**, 91–104.

Schwartz, L. W. 1974 Computer extension and analytic continuation of Stokes' expansion for gravity waves. *J. Fluid Mech.* **62**, 553–578.

Smith, R. 1975 The reflection of short gravity waves on a non-uniform current. *Math. Proc. Camb. Phil. Soc.* **78**, 517–525.

Smith, R. 1976 Giant waves. *J. Fluid Mech.* **77**, 417–431.

Stiassnie, M. & Dagan, G. 1979 Partial reflection of water waves by non-uniform adverse currents. *J. Fluid Mech.* **92**, 119–129.

Whitham, G. B. 1965 A general approach to linear and non-linear dispersive waves using a Lagrangian. *J. Fluid Mech.* **22**, 273–283.

Whitham, G. B. 1974 *Linear and Nonlinear Waves*. Wiley-Interscience.

Yue, D. K. P. & Mei, C. C. 1980 Forward diffraction of Stokes waves by a thin wedge. *J. Fluid Mech.* **99**, 33–52.